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N° 7578

March 2011

Domaine 1

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--7578--FR+ENG

INSTITUT NATIONAL
DE RECHERCHE
EN INFORMATIQUE
ET EN AUTOMATIQUE

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Domaine : Mathématiques appliquées, calcul et simulation
Équipe-Projet NACHOS

Rapport de recherche n° 7578 — March 2011 — 17 pages

Abstract: Discontinuous Galerkin (DG) methods have been the subject of numerous research activities in the last 15 years and have been successfully developed for various physical contexts modeled by elliptic, mixed hyperbolic-parabolic and hyperbolic systems of PDEs. One major drawback of high order DG methods is their intrinsic cost due to the very large number of globally coupled degrees of freedom as compared to classical high order conforming finite element methods. Different attempts have been made in the recent past to improve this situation and one promising strategy has been recently proposed by Cockburn *et al.*[CGL09] in the form of so-called hybridizable DG formulations. The distinctive feature of these methods is that the only globally coupled degrees of freedom are those of an approximation of the solution defined only on the boundaries of the elements of the discretization mesh. The present work is concerned with the study of such a hybridizable DG method for the solution of the system of Maxwell equations. In this preliminary investigation, a hybridizable DG method is proposed for the two-dimensional time-domain Maxwell equations time integrated by an implicit scheme.

Key-words: computational electromagnetics, time-domain Maxwell equations, discontinuous Galerkin method, hybridizable method.

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Une méthode Galerkine discontinue hybride implicite pour les équations de Maxwell en domaine temporel

Résumé : Les méthodes de type Galerkin discontinu (GD) ont fait l'objet de nombreux travaux ces 15 dernières années et ont été développées et appliquées avec succès dans de nombreux contextes physiques modélisés par des systèmes d'EDPs elliptiques, mixtes hyperboliques-paraboliques et hyperboliques. Un inconvénient majeur des méthodes GD d'ordre élevé est leur coût intrinsèque lié au nombre de degrés de liberté couplés comparativement aux méthodes éléments finis conformes d'ordre élevé classiques. Différentes stratégies ont été étudiées dans un passé récent pour corriger cette situation et une approche prometteuse a notamment été proposée par Cockburn *et al.*[CGL09] sous la forme de formulations GD hybrides. Une caractéristique importante de ces méthodes est que les seuls degrés de liberté couplés sont ceux associés à l'approximation de la solution aux interfaces entre éléments voisins du maillage de calcul. On s'intéresse ici à la mise au point d'une telle formulation GD hybride pour la résolution numérique du système d'équations de Maxwell. Dans cette étude préliminaire, une méthode GD hybride est proposée pour les équations de Maxwell en deux dimensions d'espace intégrées en temps par un schéma implicite.

Mots-clés : électromagnétisme numérique, équations de Maxwell en domaine temporel, méthode Galerkin discontinue, méthode hybride.

1 Introduction

The Discontinuous Galerkin (DG) method was introduced in 1973 by Reed and Hill to solve the neutron transport equation. From this time to the 90's a review of the DG methods would likely fit into one page. While it keeps almost all the advantages of the finite element method (large spectrum of applications, complex geometries, etc.), the DG method has other properties which explain the renewed interest it gains in various domains in scientific computing as witnessed by books or special issues of journals dedicated to this method [CKS00, CS05, Daw06, HW08, Riv08]:

- it is adapted to a high order approximation of the unknown field. Moreover, one may increase the degree of the approximation in the whole mesh as easily as for spectral methods but, with a DG method, this can also be done locally. In most cases, the approximation relies on a polynomial interpolation method but the DG method also offers the flexibility of applying local approximation strategies that best fit to the intrinsic features of the modeled physical phenomena (see for instance [HP09] for the Helmholtz equation in 2D).

- for unsteady problems, when the discretization in space is coupled to an explicit time integration method, the DG method leads to a block diagonal mass matrix independently of the form of the local approximation (*e.g.* the type of the polynomial interpolation). This is a striking difference with classical conforming finite element formulations. Moreover, the mass matrix is diagonal if an orthogonal polynomial basis is chosen for the discretization in each element. Besides, it is flexible with regards to the choice of the time-stepping scheme.

- it is well suited to the use of unstructured meshes and in particular of locally refined meshes with hanging nodes [Fah09a, Fah09b]. This property makes the DG method suitable to the design of a *hp*-adaptive solution strategy (*i.e.* where the characteristic mesh size h and the interpolation degree p change locally wherever it is needed).

- it is adapted to parallel computing. The compact nature of DG discretization schemes is in favor of a high computation to communication ratio especially when the interpolation order is increased.

Concerning the system of Maxwell's equations, DG methods have been developed for time-transient problems [HW02, FLLP05, CFP06] as well as for time-harmonic problems [HPS04, HPSS05, DFLP08] among others. Despite the achievements so far, it seems that a major limitation to a wider adoption of DG methods for large-scale applications in various physical domains and especially for electromagnetic wave propagation problems, is their excessive overhead in terms of computational time and memory occupancy. Several strategies have been considered to improve this situation and one promising approach for steady problems or unsteady problems time integrated implicitly has been recently proposed by Cockburn *et al.*[CGL09] in the form of hybridized DG methods. Such DG formulations rely on the introduction of a so-called *conservativity condition* at the interface between neighboring elements of the underlying discretization mesh. This condition amounts to imposing a certain transmission condition whose definition depends on the considered system of partial differential equations. At the discrete level, a single-valued approximate trace is introduced in the form of a numerical flux at inter-element boundaries. Then, *the only globally coupled degrees of freedom of the resulting DG methods are those associated to this numerical trace*. This setting results in an efficient implementation of these methods as demonstrated in [CGL09]

in the case of a model second order elliptic problem. Then, hybridized DG methods have been developed for linear and non-linear convection-diffusion equations time integrated implicitly [NPC09a, NPC09b], for Stokes flows [CG09, NPC10] and for linear elasticity problems [SCS09].

In this paper we study the formulation of a hybridized DG method for the system of Maxwell's equations; in this direction, a hybridized DG method has recently been proposed in [CHSW10] where the magneto-quasistatic approximation of the Maxwell equation is considered. In the present work, we consider the system of the two-dimensional time-transient equations integrated in time using an implicit scheme *i.e.* the starting-point of this study is the implicit DG time-transient method previously introduced in [CDL10]. This work is a preliminary step towards the development of such a hybridized DG method for the discretization of the time-harmonic Maxwell equations in order to improve the overall efficiency of the DG methods considered in [DLP07, DFLP08].

2 Problem statement and notations

2.1 The system of 3D Maxwell equations

We consider the system of Maxwell's equations in three dimensions for heterogeneous linear isotropic media with no source:

$$\varepsilon \partial_t \mathbf{E} - \mathbf{curl}\, \mathbf{H} = 0, \quad \mu \partial_t \mathbf{H} + \mathbf{curl}\, \mathbf{E} = 0, \tag{1}$$

where the symbol ∂_t denotes a time derivative, $\mathbf{E}(\mathbf{x},t) = \begin{pmatrix} E_x & E_y & E_z \end{pmatrix}^T$ and $\mathbf{H}(\mathbf{x},t) = \begin{pmatrix} H_x & H_y & H_z \end{pmatrix}^T$ are respectively the electric and magnetic fields (the symbol ^{T} denotes the transposition). These equations are set on a bounded polyhedral domain Ω of $\mathbb{R}^3$. The dielectric permittivity tensor ε and the magnetic permeability tensor μ are varying in space, time-invariant and both positive functions. Our goal is to solve system (1) in a domain Ω with boundary $\partial\Omega = \Gamma_a \cup \Gamma_m$, where we impose the following boundary conditions:

$$\begin{cases} \mathbf{n} \times \mathbf{E} = 0 \text{ on } \Gamma_m, \\ \mathbf{n} \times \mathbf{E} - \sqrt{\dfrac{\mu}{\varepsilon}} \mathbf{n} \times (\mathbf{H} \times \mathbf{n}) = \mathbf{n} \times \mathbf{E}^{\text{inc}} - \sqrt{\dfrac{\mu}{\varepsilon}} \mathbf{n} \times (\mathbf{H}^{\text{inc}} \times \mathbf{n}) \text{ on } \Gamma_a. \end{cases} \tag{2}$$

Here $\mathbf{n}$ denotes the unit outward normal to $\partial\Omega$ and $(\mathbf{E}^{\text{inc}}, \mathbf{H}^{\text{inc}})$ is a given incident field. The first boundary condition is called *metallic* (referring to a perfectly conducting surface) while the second condition is called *absorbing* and takes here the form of the Silver-Müller condition which is a first order approximation of the exact absorbing boundary condition. This absorbing condition is applied on Γ_a which represents an artificial truncation of the computational domain. Finally, system (1) is supplemented with initial conditions: $\mathbf{E}_0(\mathbf{x}) = \mathbf{E}(\mathbf{x}, 0)$ and $\mathbf{H}_0(\mathbf{x}) = \mathbf{H}(\mathbf{x}, 0)$.

2.2 The system of 2D Maxwell's equations

The 2D transverse magnetic Maxwell's equations are obtained by setting $H_x = H_x(x,y,t)$, $H_y = H_y(x,y,t)$, $H_z = E_x = E_y = 0$, $E_z = E_z(x,y,t)$ and by defining:

$$\mathbf{curl}\, E_z = \begin{pmatrix} \partial_y E_z & -\partial_x E_z \end{pmatrix}^T, \quad \text{curl}\, \mathbf{H} = \partial_x x H_y - \partial_y H_x.$$

while the boundary conditions are given by:

$$\begin{cases} E_z = 0 \text{ on } \Gamma_m, \\ E_z + \sqrt{\dfrac{\mu}{\varepsilon}}(\mathbf{n}\times\mathbf{H}) = E_z^{\text{inc}} + \sqrt{\dfrac{\mu}{\varepsilon}}(\mathbf{n}\times\mathbf{H}^{\text{inc}}) = g^{\text{inc}} \text{ on } \Gamma_a, \end{cases} \tag{3}$$

where the cross-product $\mathbf{u}\times\mathbf{v}$ in 2D is equal to the determinant $(\mathbf{u},\mathbf{v}) = u_x v_y - u_y v_x$.

2.3 Time integration

In this study, we consider the situation where system (1) is time integrated by a Crank-Nicolson scheme:

$$\begin{cases} \varepsilon\left(\dfrac{\mathbf{E}^{n+1}-\mathbf{E}^n}{\Delta t}\right) - \mathbf{curl}\left(\dfrac{\mathbf{H}^{n+1}+\mathbf{H}^n}{2}\right) = 0, \\ \mu\left(\dfrac{\mathbf{H}^{n+1}-\mathbf{H}^n}{\Delta t}\right) + \mathbf{curl}\left(\dfrac{\mathbf{E}^{n+1}+\mathbf{E}^n}{2}\right) = 0, \end{cases} \tag{4}$$

that we rewrite as:

$$\begin{cases} \dfrac{2\varepsilon}{\Delta t}\mathbf{E}^{n+1} - \mathbf{curl}\,\mathbf{H}^{n+1} = \dfrac{2\varepsilon}{\Delta t}\mathbf{E}^n + \mathbf{curl}\,\mathbf{H}^n, \\ \dfrac{2\mu}{\Delta t}\mathbf{H}^{n+1} + \mathbf{curl}\,\mathbf{E}^{n+1} = \dfrac{2\mu}{\Delta t}\mathbf{H}^n - \mathbf{curl}\,\mathbf{E}^n. \end{cases} \tag{5}$$

For the sequel, to simplify the notations, we omit the superscript $^{n+1}$. In addition, we set $\bar{\varepsilon} = \dfrac{2\varepsilon}{\Delta t}$, $\bar{\mu} = \dfrac{2\mu}{\Delta t}$, $\mathbf{b}_E = \dfrac{2\varepsilon}{\Delta t}\mathbf{E}^n + \mathbf{curl}\,\mathbf{H}^n$ and $\mathbf{b}_H = \dfrac{2\mu}{\Delta t}\mathbf{H}^n - \mathbf{curl}\,\mathbf{E}^n$. Then, the system to be solved writes as:

$$\bar{\varepsilon}\mathbf{E} - \mathbf{curl}\,\mathbf{H} = \mathbf{b}_E, \quad \bar{\mu}\mathbf{H} + \mathbf{curl}\,\mathbf{E} = \mathbf{b}_H. \tag{6}$$

2.4 Notations and approximation spaces

From now on we place ourselves in the 2D case and we consider a partition $\mathcal{T}_h$ of $\Omega \in \mathbb{R}^2$ into a set of triangles. The unit outward normal to an element K is denoted $\mathbf{n}_K$. Each non-empty intersection of the closure of two elements K_1 and K_2 is called an interface. We denote by $\mathcal{F}_h^I$ the union of all interior interfaces of $\mathcal{T}_h$, by $\mathcal{F}_h^B$ the union of all boundary interfaces of $\mathcal{T}_h$, and by $\mathcal{F}_h$ the union of $\mathcal{F}_h^I$ and $\mathcal{F}_h^B$. Furthermore, we identify $\mathcal{F}_h^B$ to $\partial\Omega$ since it is assumed that Ω is a polyhedron. Let $\mathbb{P}_p(\omega)$ denotes the space of polynomial functions of degree at most p on a domain ω. For any element $K \in \mathcal{T}_h$ we denote by $V^p(K)$ and $\mathbf{V}^p(K)$ respectively $\mathbb{P}_p(K)$ and $(\mathbb{P}_p(K))^2$. We introduce the discontinuous finite element spaces:

$$V_h^p = \left\{ v \in L^2(\Omega) \mid v|_K \in V^p(K),\ \forall K \in \mathcal{T}_h \right\},$$
$$\mathbf{V}_h^p = \left\{ \mathbf{v} \in (L^2(\Omega))^2 \mid \mathbf{v}|_K \in \mathbf{V}^p(K),\ \forall K \in \mathcal{T}_h \right\},$$

where $L^2(\Omega)$ is the space of square integrable functions on the domain Ω. In addition, we introduce a traced finite element space whose definition takes into account the metallic boundary condition (first

relation of (3)):

$$M_h^p = \left\{\eta \in L^2(\mathcal{F}_h) \mid \eta|_f \in \mathbb{P}_p(f),\ \forall f \in \mathcal{F}_h \text{ and } \eta|_{\Gamma_m} = 0\right\}. \tag{7}$$

One can note that M_h^p consists of functions which are continuous inside a triangle in 3D (or an edge in 2D), and discontinuous at its borders. For vector functions $\mathbf{v}$ and $\mathbf{w}$ in $(L^2(\omega))^2$, we denote $(\mathbf{v},\mathbf{w})_\omega = \displaystyle\int_\omega \mathbf{v}\cdot\mathbf{w}$ while for functions v and w in $L^2(\omega)$ we denote $(v,w)_\omega = \displaystyle\int_\omega vw$ if ω is a domain in $\mathbb{R}^2$ and $< v, w >_\omega = \displaystyle\int_\omega vw$ if ω is a domain in $\mathbb{R}$. We finally introduce:

$$(\cdot,\cdot)_{\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h} (\cdot,\cdot)_K,\ < \cdot,\cdot >_{\partial\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h} < \cdot,\cdot >_{\partial K},\ < \cdot,\cdot >_{\mathcal{F}_h} = \sum_{f\in\mathcal{F}_h} < \cdot,\cdot >_f .$$

We will also use for taking into account the absorbing boundary condition (second relation of (3)):

$$< \cdot,\cdot >_{\Gamma_a} = \sum_{f\in\mathcal{F}_h\cap\Gamma_a} < \cdot,\cdot >_f .$$

3 The 2D case

3.1 Classical DG formulation

In the 2D case, the system to solve writes:

$$\bar{\varepsilon}E - \operatorname{curl}\mathbf{H} = b_E, \quad \bar{\mu}\mathbf{H} + \mathbf{curl}\, E = \mathbf{b}_H, \tag{8}$$

where $E = E_z$ and $\mathbf{H} = \begin{pmatrix} H_x & H_y \end{pmatrix}^T$. Following the DG approach, approximations of the electric and magnetic fields $(E_h,\ \mathbf{H}_h)$ are sought in $V_h^p \times \mathbf{V}_h^p$ in order to satisfy approximately for all K in $\mathcal{T}_h$:

$$\begin{cases} (\bar{\varepsilon}E_h, v)_K - (\operatorname{curl}\mathbf{H}_h, v)_K = (b_E, v)_K, & \forall v \in V^p(K), \\ (\bar{\mu}\mathbf{H}_h, \mathbf{v})_K + (\mathbf{curl}\, E_h, \mathbf{v})_K = (\mathbf{b}_H, \mathbf{v})_K, & \forall \mathbf{v} \in \mathbf{V}^p(K). \end{cases} \tag{9}$$

Green's formula makes appear boundary terms which are replaced by *numerical traces* $\widehat{E}_h$ and $\widehat{\mathbf{H}}_h$ in order to ensure the connection between element-wise solutions and global consistency of the discretization. This leads to the formulation:

$$\begin{cases} (\bar{\varepsilon}E_h, v)_K - (\mathbf{H}_h, \mathbf{curl}\, v)_K - < \mathbf{n}\times\widehat{\mathbf{H}}_h, v >_{\partial K} = (b_E, v)_K, & \forall v \in V^p(K), \\ (\bar{\mu}\mathbf{H}_h, \mathbf{v})_K + (E_h, \operatorname{curl}\mathbf{v})_K - < \widehat{E}_h, \mathbf{n}\times\mathbf{v} >_{\partial K} = (\mathbf{b}_H, \mathbf{v})_K, & \forall \mathbf{v} \in \mathbf{V}^p(K). \end{cases} \tag{10}$$

Note that $\mathbf{v}\times\mathbf{n} = \mathbf{t}\cdot\mathbf{v}$ with $\mathbf{t}\times\mathbf{n} = 1$[1] where $\mathbf{t}$ is the unitary tangent vector to the face ∂K. The numerical traces $\widehat{E}_h$ and $\widehat{\mathbf{H}}_h$ state the couplings between neighboring elements. Indeed, let $f = \overline{K^-} \cap \overline{K^+} \in \mathcal{F}_h^I$, let $(\mathbf{v}^\pm,\ v^\pm)$ be the traces of $(\mathbf{v}\ ,\ v)$ on f from the interior of $K^\pm$. The numerical traces are defined as:

$$\widehat{E}_h = \{E_h\}_f + \alpha_H[\![\mathbf{H}]\!]_f, \quad \widehat{\mathbf{H}}_h = \{\mathbf{H}_h\}_f + \alpha_E[\![E_h]\!]_f, \tag{11}$$

[1] $\mathbf{n} = \begin{pmatrix} n_x & n_y \end{pmatrix}^T$ and $\mathbf{t} = \begin{pmatrix} n_y & -n_x \end{pmatrix}^T$.

where:

$$\{v\}_f = \frac{1}{2}(v^+ + v^-), \quad [\![v]\!]_f = v^+\mathbf{t}^+ + v^-\mathbf{t}^-, \quad [\![\mathbf{v}]\!]_f = \mathbf{n}^+ \times \mathbf{v}^+ + \mathbf{n}^- \times \mathbf{v}^-.$$

Many choices are possible for the numerical traces among which $\alpha_E = \alpha_H = 0$ which corresponds to the so-called *central flux based DG method* [FLLP05], $\alpha_E = \alpha_H = \mathcal{O}(1)$ [HW02] which yields th so-called *upwind flux based DG method*, and $\alpha_E = \mathcal{O}(h^{-1})$, $\alpha_H = 0$ [CS98] which corresponds to the local DG method.

3.2 Hybridized DG formulation

3.2.1 Formulation of the global problem

Following the approach described in [CGL09], the main principles of the hybridized DG formulation are that (1) the numerical traces do not directly couple the local solutions in neighboring elements, (2) these traces depend on an *hybrid* variable lying on the interfaces of the elements and, (3) a conservativity condition has to be enforced to insure the solvability of the global problem. Then, for each interface $f \in \mathcal{F}_h$, we consider a numerical trace $\widehat{\mathbf{H}}_h$ of the form:

$$\widehat{\mathbf{H}}_h = \mathbf{H}_h + \tau_K \left(E_h - \widehat{E}_h\right)\mathbf{t} \text{ on } \partial K, \tag{12}$$

where τ_K is a *local stabilization parameter*. It amounts to considering the local discontinuous Galerkin approach [CS98] in the element K. Next, we express $\widehat{E}_h$ in terms of a *new variable* $\lambda_h \in M_h^p$ and the boundary data:

$$\widehat{E}_h = \begin{cases} \lambda_h & \text{for } f \in \mathcal{F}_h^I \\ 0 & \text{for } f \in \mathcal{F}_h^B \cap \Gamma_m \end{cases} = \lambda_h, \ \forall f \in \mathcal{F}_h \text{ by (7)}. \tag{13}$$

By adding the contributions of (10) over all the elements and enforcing the continuity of the tangential component of $\widehat{\mathbf{H}}_h$, we can formulate a problem which is to find $(E_h,\ \mathbf{H}_h,\ \lambda_h) \in V_h^p \times \mathbf{V}_y^p \times M_h^p$ such that:

$$\begin{cases} (\bar{\varepsilon}E_h, v)_{\mathcal{T}_h} - (\mathbf{H}_h, \mathbf{curl}\, v)_{\mathcal{T}_h} - < \mathbf{n} \times \widehat{\mathbf{H}}_h, v >_{\partial\mathcal{T}_h} = (b_E, v)_{\mathcal{T}_h}, & \forall v \in V_h^p, \\ (\bar{\mu}\mathbf{H}_h, \mathbf{v})_{\mathcal{T}_h} + (E_h, \operatorname{curl}\mathbf{v})_{\mathcal{T}_h} - < \lambda_h, \mathbf{n} \times \mathbf{v} >_{\partial\mathcal{T}_h} = (\mathbf{b}_H, \mathbf{v})_{\mathcal{T}_h}, & \forall \mathbf{v} \in \mathbf{V}_h^p, \\ < [\![\widehat{\mathbf{H}}_h]\!], \eta >_{\mathcal{F}_h} + < \lambda_h, \eta >_{\Gamma_a} = < f^{\text{inc}}, \eta >_{\Gamma_a}, & \forall \eta \in M_h^p, \end{cases} \tag{14}$$

where for the last equation, we implicitly assumed that $\sqrt{\mu/\varepsilon} = 1$ in the neighborhood of Γ_a. The last equation of (14) which can be detailed as:

$$\sum_{f \in \mathcal{F}_h} \int_f [\![\widehat{\mathbf{H}}_h]\!]_f \eta + \sum_{f \in \mathcal{F}_h \cap \Gamma_a} \int_f \lambda_h \eta = \sum_{f \in \mathcal{F}_h \cap \Gamma_a} \int_f f^{\text{inc}} \eta, \quad \forall \eta \in M_h^p, \tag{15}$$

is called the *conservativity condition* in [CGL09]. Note that λ_h is uniquely defined over each edge since it belongs to the space M_h^p. Moreover, due to the discontinuous nature of both V_h^p and $\mathbf{V}_h^p$, the first two equations of (14) can be used to eliminate (locally) E_h and $\mathbf{H}_h$ to obtain a weak formulation in terms of

λ_h only as will be shown latter. If we insert (12) into (14), we obtain that $(E_h,\ \mathbf{H}_h,\ \lambda_h) \in V_h^p \times \mathbf{V}_h^p \times M_h^p$ is the solution of:

$$\left\{\begin{array}{rl} (\bar{\varepsilon}E_h, v)_{\mathcal{T}_h} - (\mathbf{H}_h, \mathbf{curl}\, v)_{\mathcal{T}_h} - <\mathbf{n}\times\mathbf{H}_h, v>_{\partial\mathcal{T}_h} + & \\ \qquad <\tau E_h, v>_{\partial\mathcal{T}_h} - <\tau\lambda_h, v>_{\partial\mathcal{T}_h} = (b_E, v)_{\mathcal{T}_h}, & \forall v \in V_h^p, \\ (\bar{\mu}\mathbf{H}_h, \mathbf{v})_{\mathcal{T}_h} + (E_h, \mathrm{curl}\,\mathbf{v})_{\mathcal{T}_h} - <\lambda_h, \mathbf{n}\times\mathbf{v}>_{\partial\mathcal{T}_h} = (\mathbf{b}_H, \mathbf{v})_{\mathcal{T}_h}, & \forall \mathbf{v} \in \mathbf{V}_h^p, \\ \qquad <[\![\widehat{\mathbf{H}}_h]\!], \eta>_{\mathcal{F}_h} + <\lambda_h, \eta>_{\Gamma_a} = <f^{\mathrm{inc}}, \eta>_{\Gamma_a}, & \forall \eta \in M_h^p. \end{array}\right. \tag{16}$$

For η in M_h^p and an interior edge f we have that:

$$\begin{aligned} <[\![\widehat{\mathbf{H}}_h]\!], \eta>_f &= <[\![(\mathbf{H}_h + \tau\,(E_h - \lambda_h)\,\mathbf{t})]\!], \eta>_f \\ &= <\mathbf{n}^+\times\mathbf{H}^+, \eta>_{\partial K^+} + <\mathbf{n}^-\times\mathbf{H}^-, \eta>_{\partial K^-} \\ &\quad - <\tau_{K^+}E^+, \eta>_{\partial K^+} - <\tau_{K^-}E^-, \eta>_{\partial K^-} \\ &\quad + <\tau_{K^+}\lambda_h, \eta>_{\partial K^+} + <\tau_{K^-}\lambda_h, \eta>_{\partial K^-}, \end{aligned} \tag{17}$$

and thus:

$$<[\![\widehat{\mathbf{H}}_h]\!], \eta>_{\mathcal{F}_h} = <\mathbf{n}\times\mathbf{H}_h, \eta>_{\partial\mathcal{T}_h} - <\tau E_h, \eta>_{\partial\mathcal{T}_h} + <\tau\lambda_h, \eta>_{\partial\mathcal{T}_h}\ . \tag{18}$$

Applying Green's formula in the first equation of (16) allows to rewrite this equation as:

$$(\bar{\varepsilon}E_h, v)_{\mathcal{T}_h} - (\mathrm{curl}\,\mathbf{H}_h, v)_{\mathcal{T}_h} + <\tau E_h, v>_{\partial\mathcal{T}_h} - <\tau\lambda_h, v>_{\partial\mathcal{T}_h} = (b_E, v)_{\mathcal{T}_h}. \tag{19}$$

The second equation of (16) is unchanged:

$$(\bar{\mu}\mathbf{H}_h, \mathbf{v})_{\mathcal{T}_h} + (E_h, \mathrm{curl}\,\mathbf{v})_{\mathcal{T}_h} - <\lambda_h, \mathbf{n}\times\mathbf{v}>_{\partial\mathcal{T}_h} = (\mathbf{b}_H, \mathbf{v})_{\mathcal{T}_h}. \tag{20}$$

Finally, taking into account (18), the third equation of (16) becomes:

$$\begin{aligned} <\mathbf{n}\times\mathbf{H}_h, \eta>_{\partial\mathcal{T}_h} - <\tau E_h, \eta>_{\partial\mathcal{T}_h} + <\tau\lambda_h, \eta>_{\partial\mathcal{T}_h} & \\ + <\lambda_h, \eta>_{\Gamma_a} = <f^{\mathrm{inc}}, \eta>_{\Gamma_a}\ . & \end{aligned} \tag{21}$$

3.2.2 Characterization of the reduced size problem

For α in M_h^p, the pair $(E_h^\alpha,\ \mathbf{H}_h^\alpha)$ defined on Ω, whose restriction to an element K of $\mathcal{T}_h$ is the solution to the local problem:

$$\left\{\begin{array}{l} (\bar{\varepsilon}E_h^\alpha, v)_K - (\mathbf{H}_h^\alpha, \mathbf{curl}\, v)_K - <\mathbf{n}\times\widehat{\mathbf{H}}_h^\alpha, v>_{\partial K} = (b_E, v)_K, \\ (\bar{\mu}\mathbf{H}_h^\alpha, \mathbf{v})_K + (E_h^\alpha, \mathrm{curl}\,\mathbf{v})_K - <\alpha, \mathbf{n}\times\mathbf{v}>_{\partial K} = (\mathbf{b}_H, \mathbf{v})_K, \end{array}\right. \tag{22}$$

for all $v \in V^p(K)$, and $\mathbf{v} \in \mathbf{V}^p(K)$. Note that $(E_h^{\lambda_h},\ \mathbf{H}_h^{\lambda_h},\ \lambda_h)$ is the solution to (16). If we apply Green's formula to both equations of (22) and sum over all the elements of $\mathcal{T}_h$, we obtain:

$$\left\{\begin{array}{l} (\bar{\varepsilon}E_h^\alpha, v)_{\mathcal{T}_h} - (\mathrm{curl}\,\mathbf{H}_h^\alpha, v)_{\mathcal{T}_h} - <\mathbf{n}\times(\widehat{\mathbf{H}}_h^\alpha - \mathbf{H}_h^\alpha), v>_{\partial\mathcal{T}_h} = (b_E, v)_{\mathcal{T}_h}, \\ (\bar{\mu}\mathbf{H}_h^\alpha, \mathbf{v})_{\mathcal{T}_h} + (\mathbf{curl}\, E_h^\alpha, \mathbf{v})_{\mathcal{T}_h} - <\alpha - E_h^\alpha, \mathbf{n}\times\mathbf{v}>_{\partial\mathcal{T}_h} = (\mathbf{b}_H, \mathbf{v})_{\mathcal{T}_h}, \end{array}\right.$$

for all $v \in V_h^p$, and $\mathbf{v} \in \mathbf{V}_h^p$. It leads to relations between *jumps* and *local residuals*:

$$\left\{ \begin{array}{l} (\bar{\varepsilon} E_h^\alpha, v)_{\mathcal{T}_h} - (\mathrm{curl}\, \mathbf{H}_h^\alpha, v)_{\mathcal{T}_h} - < 1, [\![(\widehat{\mathbf{H}}_h^\alpha - \mathbf{H}_h^\alpha) v]\!] >_{\mathcal{F}_h} = (b_E, v)_{\mathcal{T}_h}, \\ (\bar{\mu} \mathbf{H}_h^\alpha, \mathbf{v})_{\mathcal{T}_h} + (\mathbf{curl}\, E_h^\alpha, \mathbf{v})_{\mathcal{T}_h} - < 1, [\![(\alpha - E_h^\alpha) \mathbf{v}]\!] >_{\mathcal{F}_h} = (\mathbf{b}_H, \mathbf{v})_{\mathcal{T}_h}, \end{array} \right. \tag{23}$$

for all $v \in V_h^p$, and $\mathbf{v} \in \mathbf{V}_h^p$. We can then write more explicitly the bilinear form for the problem concerning λ_h. Indeed:

$$\begin{aligned} < [\![\widehat{\mathbf{H}}_h^\alpha]\!], \eta >_{\mathcal{F}_h} &= < [\![\mathbf{H}_h^\alpha]\!], \eta >_{\mathcal{F}_h} + < [\![\widehat{\mathbf{H}}_h^\alpha - \mathbf{H}_h^\alpha]\!], \eta >_{\mathcal{F}_h} \\ &= (\bar{\mu} \mathbf{H}_h^\alpha, \mathbf{H}_h^\eta)_{\mathcal{T}_h} + (E_h^\eta, \mathrm{curl}\, \mathbf{H}_h^\alpha)_{\mathcal{T}_h} - (\mathbf{b}_H, \mathbf{H}_h^\eta)_{\mathcal{T}_h} \\ &\quad + < [\![\widehat{\mathbf{H}}_h^\alpha - \mathbf{H}_h^\alpha]\!], \eta >_{\mathcal{F}_h}, \text{ by using the second equation of (22)} \\ &\qquad \text{summed over all } K \text{ in } \mathcal{T}_h, \\ &= (\bar{\mu} \mathbf{H}_h^\eta, \mathbf{H}_h^\alpha)_{\mathcal{T}_h} + (\bar{\varepsilon} E_h^\alpha, E_h^\eta)_{\mathcal{T}_h} \\ &\quad - < 1, [\![(\widehat{\mathbf{H}}_h^\alpha - \mathbf{H}_h^\alpha) E_h^\eta]\!] >_{\mathcal{F}_h} + < [\![\widehat{\mathbf{H}}_h^\alpha - \mathbf{H}_h^\alpha]\!], \eta >_{\mathcal{F}_h} \\ &\quad - (\mathbf{b}_H, \mathbf{H}_h^\eta)_{\mathcal{T}_h} - (b_E, b_E^\eta)_{\mathcal{T}_h}, \text{ by using the second} \\ &\qquad \text{equation of (23).} \end{aligned}$$

Thus, we deduce from the third relation of (16) that λ_h is the solution of the following *reduced size problem*:

$$a_h(\lambda_h, \eta) = b_h(\eta), \ \forall \eta \in M_h^p, \tag{24}$$

with:

$$\begin{aligned} a_h(\alpha, \eta) &= (\bar{\mu} \mathbf{H}_h^\eta, \mathbf{H}_h^\alpha)_{\mathcal{T}_h} + (\bar{\varepsilon} E_h^\alpha, E_h^\eta)_{\mathcal{T}_h} \\ &\quad + < 1, [\![(\widehat{\mathbf{H}}_h^\alpha - \mathbf{H}_h^\alpha)(\eta - E_h^\eta)]\!] >_{\mathcal{F}_h} + < \alpha, \eta >_{\Gamma_a}, \\ b_h(\eta) &= (\mathbf{b}_H, \mathbf{H}_h^\eta)_{\mathcal{T}_h} + (b_E, b_E^\eta)_{\mathcal{T}_h} + < f^{\mathrm{inc}}, \eta >_{\Gamma_a}. \end{aligned} \tag{25}$$

Moreover, using (12), we have also:

$$\begin{aligned} a_h(\alpha, \eta) &= (\bar{\mu} \mathbf{H}_h^\eta, \mathbf{H}_h^\alpha)_{\mathcal{T}_h} + (\bar{\varepsilon} E_h^\alpha, E_h^\eta)_{\mathcal{T}_h} \\ &\quad + < 1, [\![\tau (E_h^\alpha - \alpha)(E_h^\eta - \eta)]\!] >_{\mathcal{F}_h} \\ &\quad + < \alpha, \eta >_{\Gamma_a}, \end{aligned} \tag{26}$$

and the bilinear form for λ_h is symmetric positive (we check that it is also definite in the following).

3.2.3 Well-posedness of the local solver

Considering the local problem (22), with $\alpha = 0$, $\mathbf{b}_H = 0$, and $b_E = 0$, taking $\mathbf{v} = \mathbf{H}_h^\alpha$, $v = E_h^\alpha$ and using (12) in the first equation of (22):

$$\begin{aligned} (\bar{\varepsilon} E_h^\alpha, E_h^\alpha)_K - (\mathbf{H}_h^\alpha, \mathbf{curl}\, E_h^\alpha)_K - < \mathbf{n} \times \mathbf{H}_h^{\mathrm{m}}, E_h^{\mathrm{m}} >_{\partial K} \\ - < \tau_K E_h^\alpha, E_h^\alpha >_{\partial K} - < \tau_K \alpha, E_h^\alpha >_{\partial K} = 0. \end{aligned} \tag{27}$$

Then, using:

$$(\mathbf{H}_h^\alpha, \mathbf{curl}\, E_h^\alpha)_K + < \mathbf{n} \times \mathbf{H}_h^\alpha, E_h^\alpha >_{\partial K} = (\mathrm{curl}\, \mathbf{H}_h^\alpha, E_h^\alpha)_K,$$

allows to rewrite (27) as:

$$\begin{aligned}(\bar{\varepsilon} E_h^\alpha, E_h^\alpha)_K - (\mathrm{curl}\, \mathbf{H}_h^\alpha, E_h^\alpha)_K + < \tau_K E_h^\alpha, E_h^\alpha >_{\partial K} \\ - < \tau_K \alpha, E_h^\alpha >_{\partial K} = 0.\end{aligned} \tag{28}$$

Adding the second equation of (22) with (28) leads to:

$$\begin{aligned}(\bar{\varepsilon} E_h^\alpha, E_h^\alpha)_K + (\bar{\mu} \mathbf{H}_h^\alpha, \mathbf{H}_h^\alpha)_K + < \tau_K E_h^\alpha, E_h^\alpha >_{\partial K} \\ = < \tau_K \alpha, E_h^\alpha >_{\partial K} + < \alpha, \mathbf{n} \times \mathbf{H}_h^\alpha >_{\partial K}.\end{aligned} \tag{29}$$

For $\alpha = 0$ and assuming $\tau_K \geq 0$, (29) implies that $E_h^\alpha = 0$ and $\mathbf{H}_h^\alpha = 0$ on K since ε and μ are strictly positive real numbers.

3.3 Existence and uniqueness of the solution of the reduced size problem

Here, we prove that the reduced system (24)-(25)-(26) has a unique solution.

Theorem 1 *If the stabilization parameter satisfies the condition $\tau > 0$ then the solution λ_h of (24)-(25)-(26) exists and is unique.*

Proof. The existence and uniqueness of λ_h follows if we show that the solution $\alpha \in M_h^p$ of the problem:

$$a_h(\alpha, \eta) = 0, \quad \forall \eta \in M_h^p,$$

is $\alpha = 0$. Taking $\eta = \alpha$ in (26) yields:

$$\begin{aligned}a_h(\alpha, \alpha) = (\bar{\mu} \mathbf{H}_h^\alpha, \mathbf{H}_h^\alpha)_{\mathcal{T}_h} + (\bar{\varepsilon} E_h^\alpha, E_h^\alpha)_{\mathcal{T}_h} + < 1, [\![\tau (E_h^\alpha - \alpha)(E_h^\alpha - \alpha)]\!] >_{\mathcal{F}_h} \\ + < \alpha, \alpha >_{\Gamma_a},\end{aligned}$$

which is equivalent to::

$$\begin{aligned}a_h(\alpha, \alpha) = (\bar{\mu} \mathbf{H}_h^\alpha, \mathbf{H}_h^\alpha)_{\mathcal{T}_h} + (\bar{\varepsilon} E_h^\alpha, E_h^\alpha)_{\mathcal{T}_h} + < 1, \tau (E_h^\alpha - \alpha)(E_h^\alpha - \alpha) >_{\partial \mathcal{T}_h} \\ + < \alpha, \alpha >_{\Gamma_a},\end{aligned}$$

and finally:

$$\begin{aligned}a_h(\alpha, \alpha) = (\bar{\mu} \mathbf{H}_h^\alpha, \mathbf{H}_h^\alpha)_{\mathcal{T}_h} + (\bar{\varepsilon} E_h^\alpha, E_h^\alpha)_{\mathcal{T}_h} + < \tau (E_h^\alpha - \alpha), (E_h^\alpha - \alpha) >_{\partial \mathcal{T}_h} \\ + < \alpha, \alpha >_{\Gamma_a}.\end{aligned} \tag{30}$$

Then, a sufficient condition for having $a_h(\alpha, \alpha) = 0$ is to require that for each of the terms of (30). It is then clear that the two first terms imply that $\mathbf{H}_h^\alpha$ and E_h^α are zero on each $K \in \mathcal{T}_h$ since $\bar{\varepsilon}$ and $\bar{\mu}$ are

strictly positive real numbers. On the other hand, the last term yields that $\alpha = 0 \ \ \forall f \in \mathcal{F}_h \cap \Gamma_a$. Finally, since $E_h^\alpha = 0 \ \ \forall K \in \mathcal{T}_h$ we have that:

$$< \tau(E_h^\alpha - \alpha), (E_h^\alpha - \alpha) >_{\partial \mathcal{T}_h} = < \tau\alpha, \alpha >_{\partial \mathcal{T}_h} .$$

Imposing that $< \tau\alpha, \alpha >_{\partial \mathcal{T}_h} = 0$ and assuming $\tau > 0$ implies that $\alpha = 0$ on ∂K, $\forall K \in \mathcal{T}_h$. In summary, $\alpha = 0$ on $\mathcal{F}_h \cap \Gamma_a$ and on $\mathcal{T}_h$, therefore $\alpha = 0$ everywhere.

■

4 Numerical results

In this section we present numerical results for model 2D problems that aim at studying the convergence properties of the proposed hybridized DGTD (Discontinuous Galerkin Time Domain) method for the solution of the transverse magnetic time-domain Maxwell's equations.

4.1 Assessment of the accuracy in space

In order to assess the accuracy in space, an easier situation than the full space-time discretization is first considered. Indeed, α and β are taken equal to one, and the source term is computed in order that the solution is:

$$\begin{aligned} E_z &= \sin(\pi x)\sin(\pi y), \\ H_x &= -\pi \sin(\pi x)\cos(\pi y), \\ H_y &= \pi \cos(\pi x)\sin(\pi y). \end{aligned} \tag{31}$$

The considered mesh is structured and is regularly refined to obtain a higher accuracy for the solution of the problem. The parameter τ is taken equal to one. The obtained convergence rates for different interpolation orders are plotted in Figure 1 for the component E_z. The convergence rates for the component H_x are given in Figure 2; the behavior for H_y is similar. We observe that the convergence of the L_2-error is optimal for both fields.

4.2 Assessment of the accuracy in time

The second test problem that we consider is the propagation of an eigenmode in a closed cavity (Ω is the unit square) where the walls are metallic ($E_z = 0$ on $\partial\Omega$). The exact solution has the following form:

$$\begin{aligned} E_z &= \sin(k_y \pi x)\sin(k_x \pi y)\cos\left(\sqrt{k_x^2 + k_y^2}\pi t\right), \\ H_x &= -\frac{k_x}{\sqrt{k_x^2 + k_y^2}}\sin(k_y \pi x)\cos(k_x \pi y)\sin\left(\sqrt{k_x^2 + k_y^2}\pi t\right), \\ H_y &= \frac{k_y}{\sqrt{k_x^2 + k_y^2}}\cos(k_y \pi x)\sin(k_x \pi y)\sin\left(\sqrt{k_x^2 + k_y^2}\pi t\right). \end{aligned} \tag{32}$$

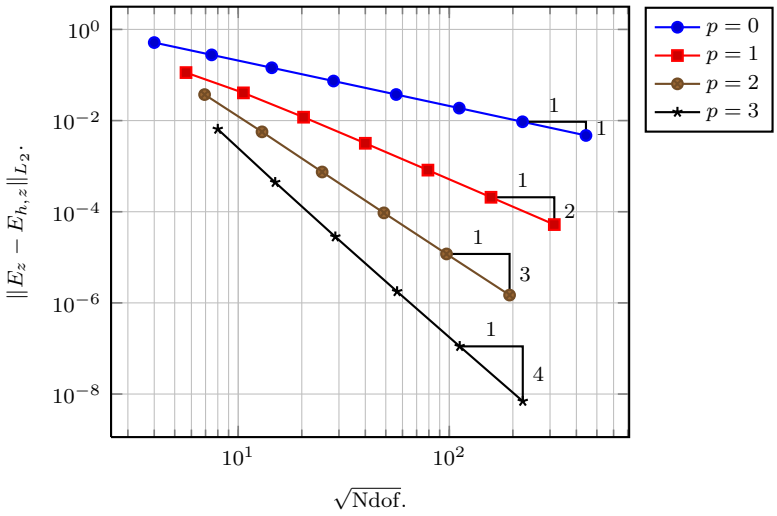

Figure 1: L_2-error for the component E_z for several polynomial approximations. Ndof is the number of degrees of freedom.

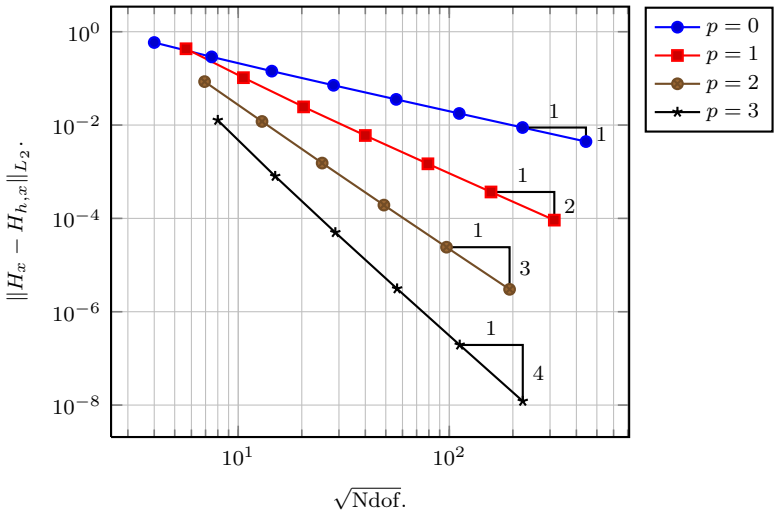

Figure 2: L_2-error for the component H_x for several polynomial approximations. Ndof is the number of degrees of freedom.

For the series of results discussd here, we have fixed $k_x = k_y = 1$. The physical simulation time has been fixed to $T = 2$. The time step is chosen as $\Delta t = \text{CFL}\Delta t_{\min}$ where $\Delta t_{\min}$ is the global minimal time step over the whole mesh. As we want to check the second order accuracy in time for the Crank-Nicholson scheme, a regular mesh with a polynomial approximation of order 2 is considered for the discretization in space; this means that the discretization error in time is predominant and we can thus vary the coefficient CFL to study the consistency error. The time evolution of the L_2-error for the component E_z is given in Figure 3. The same evolution is given for H_x in Figure 4, results for H_y are similar. A closer look at the results show that the L_2-error is divided by 4 when the time-step is divided by 2; this is illustrated in Figure 5 where the L_2-error for the component E_z at time $t = 1.5$ is interpolated (simple linear interpolation) for the four chosen values of Δt.

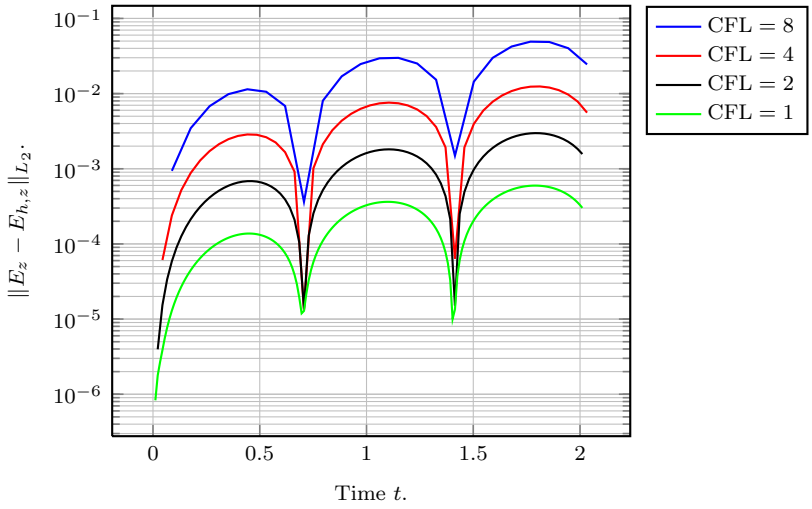

Figure 3: Time evolution of the L_2-error for the component E_z.

5 Conclusion

In this paper we have proposed a hybridized DG method for the system of 2D transverse magnetic Maxwell's equations time integrated implicitly. Starting from the implicit DGTD method previously introduced in [CDL10], we have followed the approach described in [CGL09] to obtain a hybridized DG method which is based on a hybrid variable taken to be the numerical trace of the electric field on the interface between two neighboring elements. In the present 2D setting, the electric field has one single component, therefore we anticipate a larger gain in the size of the system of globally coupled degrees of freedom in the 3D case. We have presented numerical results for model 2D problems that aim at studying the convergence properties of the proposed hybridized DGTD method. Indeed, the method demonstrates optimal convergence in L_2 norm when considering the discretization in space only. Besides, we have also

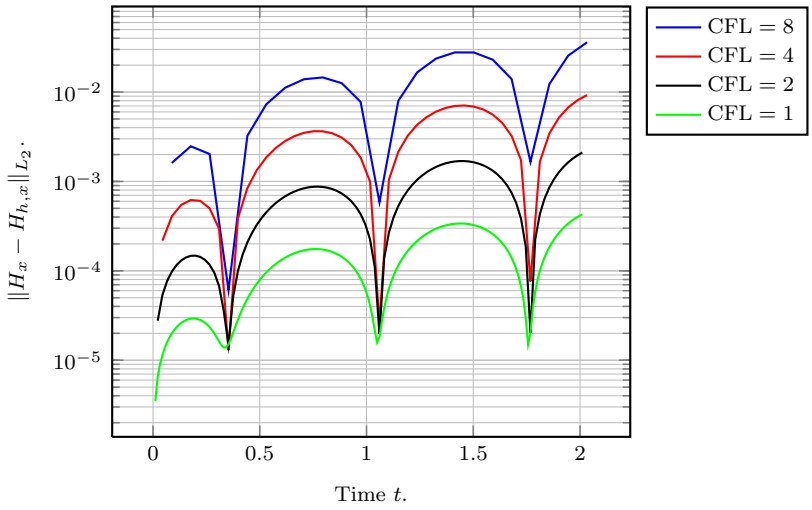

Figure 4: Time evolution of the L_2-error for the component H_x.

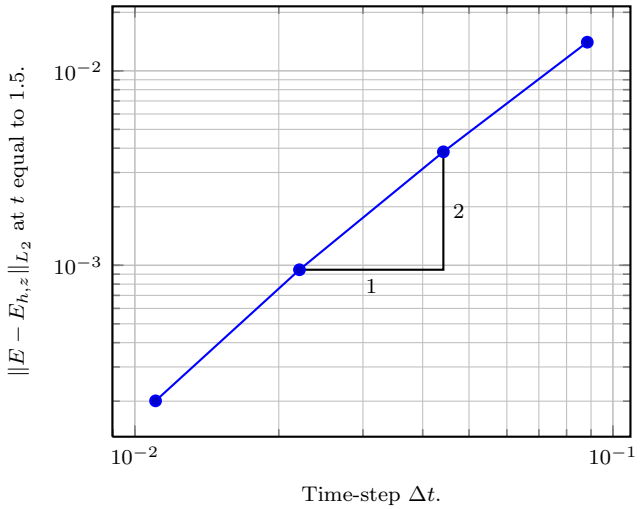

Figure 5: L_2-error on the component E_z at time $t = 1.5$ for different time-steps.

checked second order accuracy in time as can be expected from the use of a Crank-Nicholson scheme for the time integration. This work is a preliminary step towards the development of such a hybridized DG method for the discretization of the time-harmonic Maxwell equations in the 2D and 3D cases which is the subject of an ongoing study.

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ISSN 0249-6399